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2025 IMO Shortlist A1.

In a single-player game, you choose two distinct nonzero integers a and b for $x^2 + ax + b = 0$. If the equation has two distinct nonzero integer solutions u and v, then you replace the equation to $x^2 + ux + v = 0$ on the next turn. Find all initial (a, b) such that the game can be played forever.

$x^2+2x+1=0$

$u=v=-1$ ✗

$x^2-3x+2=0$

$x^2+2x+1=0$ ✗ $x^2+x+2=0$ ✗ $\Delta<0$

$u=1,\ v=-1$

$x^2-1=0$ ✗ $u\neq -v$

$u=1,\ v=-2$

$x^2+x-2=0$

$x^2+x-2=0$

⋮ Forever

$x^2+ax+b=0$

$\Downarrow$

$x^2+ux+v=0$

$\Downarrow$

$x^2+px+q=0$

$\Downarrow$

⋮ ↺ what is this??

$x^2-(\alpha+\beta)x+\alpha\beta=0$

$\Downarrow$

$x^2+\alpha x+\beta=0$

$\alpha\beta=\beta \Rightarrow \alpha=1$

$-\alpha-\beta=\alpha \Rightarrow \beta=-2$

$x^2+x-2=0$

$(1,-2)$ is unique?

$v=pq$

$|v|=|pq|\geq|q|$

↳ when does $|v|$ stop descending?

$\Rightarrow |p|=1$

If $p\neq 1$, $|v|$ descend!

$\Downarrow$

At the end, $|p|=|u|=1$

$u=-p-q$

$q=-u-p$

$q=-2$, 0, or 2

✗

$(1,-2)$

$x^2+x-2=0$

↳ our goal also our start

$x^2+x+2=0$ ✗ $\Delta<0$

$\boxed{(1,-2)}$

□